Maths Learning Service: Revision

Logarithms

Mathematics IMA

You are already familiar with some uses of powers or indices. For example:

$$\begin{aligned} 10^4 &= 10 \times 10 \times 10 \times 10 = 10,000 \\ 2^3 &= 2 \times 2 \times 2 = 8 \\ 3^{-2} &= \frac{1}{3^2} = \frac{1}{9} \end{aligned}$$

Logarithms pose a related question. The statement

$$\log_{10} 100$$

asks "what power of 10 gives us 100?" The answer is clearly 2, so we would write

$$\log_{10} 100 = 2.$$

Similarly

$$\log_{10} 10,000 = 4 \quad \text{and} \quad \log_2 8 = 3$$

In general:

$$\boxed{a^x = b \quad \Leftrightarrow \quad \log_a b = x}$$

The number appearing as the subscript of the log is called the *base* so "$\log_{10}$" is read as "logarithm to base 10". The two most common bases you will encounter are 10 and the *exponential base* $e = 2.71828$

Laws of Logarithms

Given the link between indices and logarithms, we should be able to derive laws for logarithms based on the index laws.

Consider the following argument:

The definition of a logarithm allows us to write the number A as $b^{\log_b A}$ for some base b. Similarly, we could write

$$\begin{aligned} B &= b^{\log_b B} \\ \text{and } A \times B &= b^{\log_b (A \times B)} \end{aligned} \qquad (1)$$

On the other hand, using the index laws, we get

$$A \times B = b^{\log_b A} \times b^{\log_b B} = b^{(\log_b A + \log_b B)}.$$

Comparing this expression for $A \times B$ with (1) we have

$$A \times B = b^{\log_b A + \log_b B} = b^{\log_b (A \times B)}.$$

Since the bases are the same,

$$\log_b A + \log_b B = \log_b (A \times B)$$

By similar arguments the Laws of Logarithms are as follows:

$$\boxed{\begin{aligned} \log_b A + \log_b B &= \log_b (A \times B) \\ \log_b A - \log_b B &= \log_b \left(\frac{A}{B}\right) \\ \log_b (A^n) &= n \log_b A \end{aligned}}$$

Here are a few examples where these laws can be used to solve equations.

(a) Find x such that $2\log_b 4 - 3\log_b 2 + \log_b 2 = \log_b x$.

$$\begin{aligned} \log_b (4^2) - \log_b (2^3) + \log_b 2 &= \log_b x \\ \log_b 16 - \log_b 8 + \log_b 2 &= \log_b x \\ \log_b \left(\tfrac{16}{8}\right) + \log_b 2 &= \log_b x \\ \log_b \left(\tfrac{16 \times 2}{8}\right) &= \log_b x \\ \log_b 4 &= \log_b x \\ \text{so} \quad x &= 4. \end{aligned}$$

(b) Find t such that $1000 = 100\left(2^{\frac{t}{5}}\right)$.

$$\begin{aligned}
10 &= 2^{\frac{t}{5}} \\
\log_{10} 10 &= \log_{10}\left(2^{\frac{t}{5}}\right) \qquad \text{(or any other base, such as } e\text{)} \\
1 &= \frac{t}{5}\log_{10} 2 \\
t &= \frac{5}{\log_{10} 2} \\
&= \frac{5}{0.30103} \\
&= 16.609\ldots
\end{aligned}$$

(c) In the previous example we chose $\log_{10}$ since this made $\log_{10} 10$ very easy and $\log_{10} 2$ could be found on a calculator. If we had used $\log_2$ we would have had to find $\log_2 10$, for which there is no calculator button.

It is possible to find logs to any base by noting the following argument:

$$\begin{aligned}
\text{Let } y = \log_a b \quad \Leftrightarrow \quad a^y &= b \\
\ln(a^y) &= \ln b \\
y \ln a &= \ln b \\
y &= \frac{\ln b}{\ln a}.
\end{aligned}$$

(Using $\log_{10}$ works just as well of course.) For example

$$\begin{aligned}
\log_2 8 &= \frac{\ln 8}{\ln 2} & &= \frac{\log_{10} 8}{\log_{10} 2} \\
&= \frac{2.07944\ldots}{0.69314\ldots} & &= \frac{0.9031\ldots}{0.3010\ldots} \\
&= 3 & &= 3.
\end{aligned}$$

Exercises

(4) Express as a single logarithm:

(a) $\log_b 8 - \log_b 2$ (b) $2\log_b 3 + \log_b 2$ (c) $1 - \log_{10} 4$
(d) $\log_b a + \log_b\left(\frac{1}{a}\right)$

(5) Write in terms of $\log_b 2$ and $\log_b 3$:

(a) $\log_b 6$ (b) $\log_b 8$ (c) $\log_b 24$

(6) Find, using a calculator (to 4 decimal places):

(a) $\log_2 6$ (b) $\log_3 8$ (c) $\log_3 1000$ (d) $\log_3 100,000$
(e) $\log_3 0.001$ (f) $\log_3 0.00001$ (g) $\log_3 1$

(7) Solve for x:

(a) $9 = 10\left(2^{-\frac{x}{}}\right.$

Answers to Exercises

(1) (a) 3 (b) 2 (c) 6 (d) 3 (e) 2 (f) 2

(2) (a) $x = 10^5 = 100,000$ (b) $y = 2^5 = 32$ (c) $z = 3^4 = 81$

(3) (a) 1 (b) 0 (c) -1 (d) -2 (e) 0 (f) -2

(4) (a) $\log_b 4$ (b) $\log_b 18$ (c) $\log_{10}\left(\frac{5}{2}\right)$ (d) $\log_b 1 = 0$

(5) (a) $\log_b 2 + \log_b 3$ (b) $3\log_b 2$ (c) $3\log_b 2 + \log_b 3$

(6) (a) 2.5850 (b) 1.8928 (c) 6.2877 (d) 10.4795 (e) -6.2877

(f) -10.4795 (g) 0

(7) (a) 246.245 (b) 0.01918